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Information Gain and Loss

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Technical Report

November 1998

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Technical Report
of the United States Military Academy's
Operations Research Center for Excellence

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CONTENTS

ABSTRACT

A measure of performance of battle information systems, called "information gain," has been under development by faculty and cadets at the US Military Academy. It is based on decreases (and increases) in Shannon's entropy that result from receipt of data by a tactical commander. Such data might include, for example, reconnaissance reports, movements of the opposing forces and results of engagements. We have investigated the effects of target mobility on information loss when surveillance of a target is broken. This is based on a simple stochastic model of target movements, together with an approximation appropriate for implementation in a spreadsheet. We found the shapes of information loss curves are dependent on terrain features affecting the movement of the target, and that information is lost surprisingly rapidly once contact with the target is lost.

INTRODUCTION

As a result of studies concerning the architecture of the US Army's future force, known as "Force XXI" and "Army After Next," it has become apparent that information dominance is a critical requirement (TRADOC Pamphlet 525-5 1994). Thus there is heightened interest in optimizing battlefield information systems and managing related information processes. It appears critically important to include measures of combat information in designing and evaluating combat systems and devising tactics for their use. Commonly used analytic measures are based on system throughput characteristics such as the volume or rate of messages, message quality or timeliness, system reliability, or characteristics of the data given in messages, such as detection rates. At the other end of the spectrum are the extraction of meaning from data received and its use in decision-making. The cognition of, and response to, information conveyed in a given

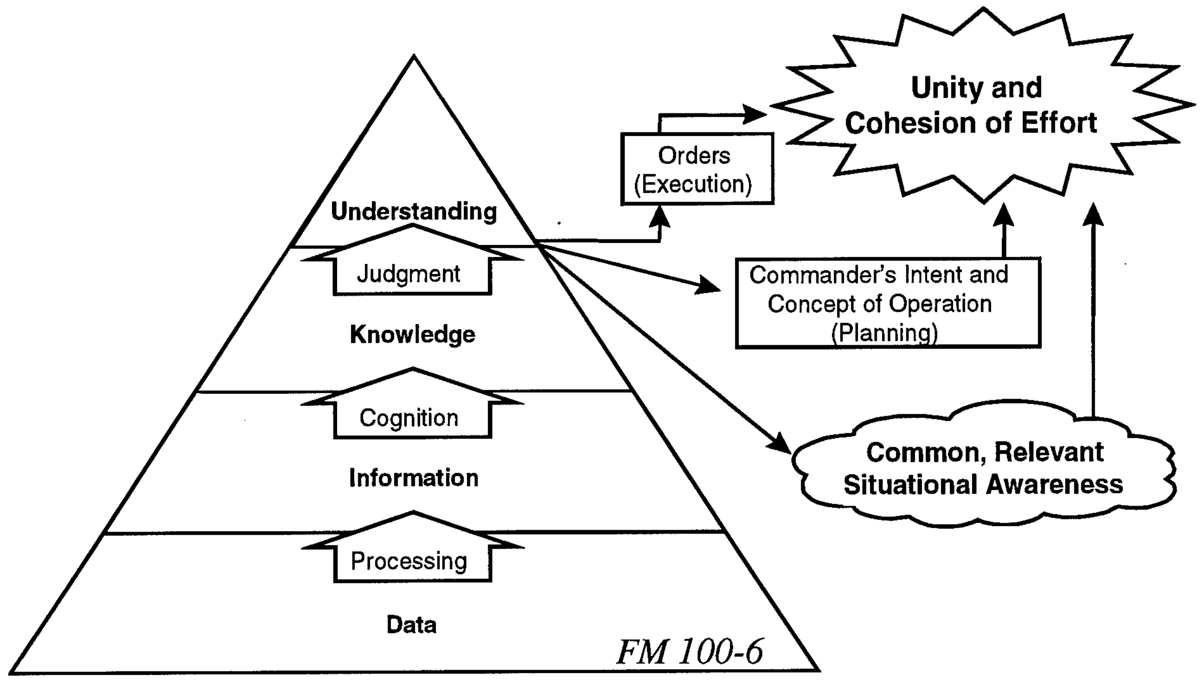

Figure 1. Situational Awareness.

set of data depends upon the receiving commander. This human process depends on the circumstances of the situation, as well as the personality, training, and experience of the commander. This is illustrated by the situational awareness pyramid (FM 100-6) shown in Figure 1.

In recent years, several faculty members and cadet groups in the Department of Systems Engineering at the US Military Academy have been developing a model of information gain. It is at a level between dealing characteristics of the physical communications system and dealing with human cognition and response of the decision-maker, about mid-way up the pyramid shown in Figure 1. The approach is to measure the level of information a commander possesses at a given point in time by modeling the amount of *uncertainty* he has about his adversary, in terms of probability distributions over sets of possible enemy states. When the commander receives data from a source such as a reconnaissance platform, the probability distributions are updated, using Bayes' formula or other means. The resultant "posterior" distribution is assumed to reflect the new state of the commander's uncertainty. The information gained as a result of the data received is measured by the decrease in Shannon's entropy from the prior to posterior distributions. This approach is not *ad hoc*; Barr and Sherrill (1996) show that, under seemingly reasonable assumptions, decrease in entropy is the unique appropriate measure of information gain. The information gain measure has been successfully applied in a variety of systems evaluations (Sherrill and Barr 1996, Marin and Barr 1997, Barr 1998).

The work reported here is focused on determining effects of target mobility on information loss. If a target is detected at a certain location, the probability mass function of that target's location amounts to a "unit-spike" of probability over the target's location. If surveillance of a mobile target is interrupted, for example if line of sight with the target is lost, the probability spike begins

to "melt," and probability "flows" to surrounding areas. But what is the precise nature of this process, and how does it relate to target movement capabilities and terrain attributes? And what is the shape of the information gain curve with this decrease? In the following sections of this report, we present results of an investigation of these questions.

1. THE INFORMATION GAIN MEASURE

In this section, we introduce the information gain measure and discuss some of its properties.

A slight extension of Shannon's development of entropy in a communications framework (Shannon 1948) provides a characterization of the information gain measure. Suppose a Blue commander's area of concern consists of a set of non-overlapping cells that may contain an enemy asset. Suppose $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is the prior probability distribution over n possible states, representing the Blue commander's uncertainty of Red's presence or location at some specific time, and suppose the uncertainty he has at some later time is represented by a posterior distribution, $\mathbf{p}^*$. Denote the information gained in resolving the uncertainty represented by $\mathbf{p}$ to that represented by $\mathbf{p}^*$, by $\delta(\mathbf{p}, \mathbf{p}^*)$. Under several reasonable assumptions about the properties of the function δ, it follows the function must be of the form

$$\delta(\mathbf{p}, \mathbf{p}^*) = \Sigma\, p^*_i \ln (p^*_i) - \Sigma\, p_i \ln(p_i) \quad (1)$$

which is just the decrease in Shannon's entropy from the prior to posterior situations. A formal statement of this result is given in (Barr and Sherrill 1996), along with some elementary properties of the function δ.

If a discrete system can be in state i with probability p_i; $i = 1,2,\ldots,n$, Shannon defined its *entropy* to be $-\Sigma\, p_i \ln(p_i)$, where the sum is over all n states and the logarithm is to the base 2, so entropy is measured in *bits*. (Since zero is not in the domain of the logarithm function, we define $0\cdot\ln(0)$ to be *0*). Entropy is a measure of the dispersion of probability mass over points, without regard to what those points are. This distinguishes entropy from common statistical measures of dispersion such as variance. If a system can be in any of n possible states, the entropy of the system can range between 0 (when the exact state of the system is known) and $\ln(n)$ bits (when the state of the system is uniformly distributed over the possible states).

A simple interpretation of values of information gain can be based on the elimination of possible states by receipt of data. Suppose, for example, initially any of n states are equally likely, and data are received showing that $(n - m)$ of the states are not possible. The posterior would then be uniform over m states, so the information gain would be $-\ln(m) + \ln(n) = \ln(n/m)$ bits. For example, if the area of a "region of uncertainty" is halved by data from a reconnaissance report, the information gain is one bit. An information gain of n bits is equivalent to the gain realized in reducing an area of uncertainty to $\frac{1}{2}^n$ its original size.

2. THREE TARGET MOTION MODELS

The posterior distribution of a target may be affected by the possible movements of the target, in addition to the receipt of data from sensors. In this section we describe three simple, yet plausible, models of target motion, and their effects on computation of changes in entropy. These models are the Square Uniform model, the Circular Uniform model, and the

Exponential Cone model. All three models assume the target moves in accordance with certain simple stochastic properties. For demonstration purposes in this paper, the commander's area of concern has been divided into equally-sized square cells. Furthermore, our examples consider only one potentially mobile target which has been positively identified in a certain location. Thus the probability of finding the target in that cell is 1.0 at the moment when visual contact with the target is broken (which we call "time-0."

a. Square Uniform Model

Let us first consider the Square Uniform model of target movement. This model is based on the assumption that, one time increment after the target was last observed ("time-1"), it is equally likely to be found in its last known cell as in any cell whose boundary touches that cell. At time-2, this probability distribution grows to a uniform distribution over the set of cells which includes the time-1 cells plus all those cells that border the time-1 cells. The effect of this algorithm is such that after n time increments, the size of the area of uncertainty has grown to a $(2n+1) \times (2n+1)$ square of cells. A uniform distribution of the target location at time-n would assign a probability of $1/(2n+1)^2$ at each of these cells. Thus, the entropy at time-n would be $2 \cdot \ln(2n+1)$.

0	0	0	0	0
0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	0
0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	0
0	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	0
0	0	0	0	0

Figure 2. Square Uniform Movement Characteristics

This model makes it easy to compute subsequent probability distributions and their associated information gains, but the target movements are not based on realistic physical movement characteristics. For example, it seems that a sequence of increasing circular shaped regions might provide a more realistic representation of target locations over time. It would seem impossible for a vehicle to travel to the extreme corners of the square in the same amount of time it could only reach the midpoint of an edge. Another weakness of the Square Uniform model is that it ignores vehicle movement speed, thus treating all mobile targets the same. Slower vehicles could cover less distance during a given time interval than faster vehicles, decreasing the number of cells to which the slower target could travel.

b. Circular Uniform Model

Let us consider a distribution that might represent more realistic movement qualities the square uniform model. In this case, we wish to limit the number of cells that could be reached in a given time by considering the potential distance, D_P, that a target could travel in the given time. One could create a circular pattern by assigning uniform probabilities to those cells within distance D_P from the time-0 cell. This model would have the effect of "chopping off" the corners of the square uniform distribution as shown in Figure 3.

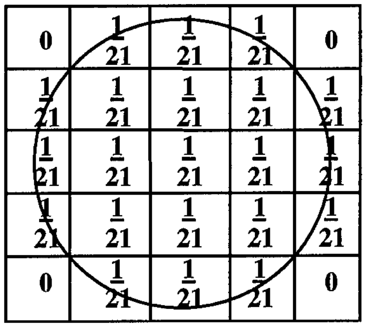

Figure 3. Circular Uniform Movement Characteristics

A uniform distribution such as this would be appropriate under certain assumptions about a target's movement abilities. Appendix A contains a discussion of a target's random speed, direction, and time of movement. In the appendix, we show that a triangular distribution of target speed for a maximum length of time along with random direction (or equivalently, a triangular distribution of movement time at maximum speed along with random direction) could create the circular uniform distribution of target location at a given time.

An algorithm to assign appropriate probabilities to cells in the region of interest can be defined as follows. After determining each cell's distance from the origin, a comparison with D_P at any given time would establish whether or not the target could theoretically occupy that cell at the given time. Cells with distance greater than D_P from the origin would have zero probability of being occupied, and cells with distance less than or equal to D_P would have equal probabilities of being occupied.

The circular uniform distribution seems to be a more realistic probability distribution than the square uniform distribution. The number of cells having positive probability with this approach occupies an area roughly 30% smaller than that defined by the uniform square model. However, the limitations imposed on the stochastic properties of target movement in order to justify the circular uniform distribution seem rather restrictive. Perhaps we could discover an approach that treats target speed, direction, and time as *independent* random variables.

c. Exponential Cone Model

Rather than declaring a location probability distribution that we think should be applied to a target's movement, let us instead make a few plausible assumptions about a target's potential movement and work towards identifying the shape of the resulting distribution. As mentioned above, the circular uniform distribution arises when either of two rather bold assumptions about a target's movement speed and time of movement are imposed. Let us examine the shape of the target location distribution based on a more plausible set of assumptions about target movement. For the purposes of this paper, we shall assume a target's movement speed S is determined in accordance with a triangular distribution, $S \sim \text{Tri}(0, a, a)$ (see Figure 4). Thus there is a movement likelihood of zero of moving at zero km/hr, increasing linearly to the greatest likelihood of moving at maximum speed a. We assume we know nothing of the target's intended direction Θ, so Θ is uniformly distributed over the set of possible directions, $\Theta \sim U(0, 2\pi)$. Similarly, we shall assume there is an equal likelihood that the target begins moving at any time T. If we normalize the elapsed time so that $t = 0$ at time-0 and $t = 1$ at the present time, then the assumption about the target's movement time can be written as $T \sim U(0, 1)$. Finally, we assume S, Θ, and T are mutually independent random variables.

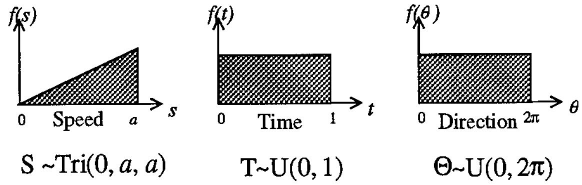

Figure 4. Assumed Movement Characteristics Probability Distributions

With these assumptions about the target's movement characteristics, the bivariate distribution of target location at time-t is given by the following joint probability density function (see the derivation in Appendix B):

$$f_{X,Y}(x,y) = \frac{1}{a\pi}\left(\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{a}\right); \quad x^2 + y^2 \leq a^2 \tag{2}$$

The marginal distribution of distance of the target from the origin is

$$f_X(x) = \frac{2}{a\pi}\left(\ln\frac{\sqrt{a^2-x^2}+a}{|x|} - \frac{\sqrt{a^2-x^2}}{a}\right) \tag{3}$$

(The vertical asymptote at the origin is not a practical issue, since the probability model does not depend on behavior of the density function at isolated points. Since the cells have positive area, one can assume the likelihood of finding the target in its original cell is given by equation (3) with argument x equal to the distance to the closest edge of that cell). Since the shape of the marginal density given in (3) is like a curved cone, we call this the "exponential cone model."

The exponential cone model seems to better represent typical target movement characteristics than do the models based on uniform distributions because we would expect there to be a greater likelihood of finding the target somewhere near cell-0. Also, intuitively, we would expect there to be a very small probability of finding the target at the maximum radius, as this would require the target to begin moving away from the origin at maximum speed at time-0. In Figure 5, we can see the exponential decay of likelihood as distance from the origin increases. Figure 5 shows that the location with greatest likelihood is the center of cell-0; the likelihood decreases monotonically to zero at the maximum radius of possible movement.

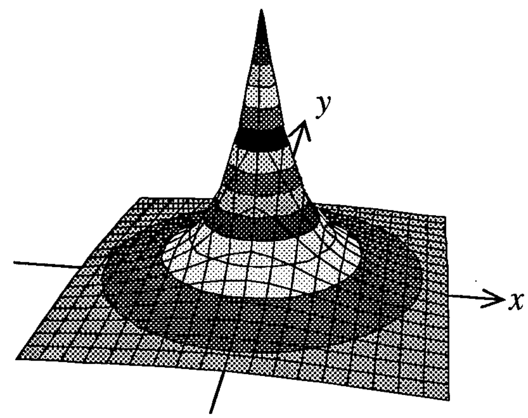

Figure 5: Plot of the marginal density of target distance from cell-0.

3. AN APPROXIMATING ALGORITHM: The Linear Cone Model

In order to investigate information loss implications of the exponential cone model, it is convenient to develop a simple algorithm that generates target location likelihoods approximately in accordance with the exponential cone distribution. We would like for this algorithm to replicate as many of the features of the Exponential Cone model as possible, yet be simple enough to allow easy implementation in a spreadsheet. Possibly the easiest way to approximate the curve shown in Figure 5 is to use a linear approximation to the marginal distribution. When this approximation is

rotated around the origin, the result is a "tent" shape which we shall call the Linear Cone model.

The Linear Cone model has a maximum value at the origin and decreases linearly to zero at the maximum possible movement radius. Although it does not exactly match the shape of the Exponential Cone model, it provides a much better approximation than does the Circular Uniform model. Using sums of squares of the differences between the Exponential Cone model and the Circular Uniform and Linear Cone models, we found that the Circular Uniform model has a coefficient of fit of 0.47, while the Linear Cone model has a 0.61 coefficient of fit to the curve shown in Figure 5. Better approximations are certainly possible, but we shall see that the Linear Cone model admits a convenient algorithm.

We can create the Linear Cone by tracking the total possible time that the target could have spent in a cell at any time after contact has been broken. The possibility of finding the target outside the radius of maximum possible movement is zero. Within this radius we shall determine the earliest time that it was possible for each cell to have been reached. Once it is possible for a cell to be occupied, it is a simple matter to subtract the earliest occupation time from the current elapsed time to find the maximum possible occupation time per cell.

If we plot the maximum occupation time values for each cell, it is seen that a "Linear Cone" shape results. Figure 6 shows an example of this idea.

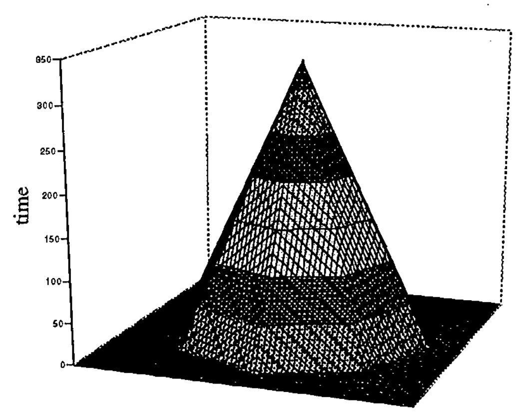

Figure 6: Elapsed Time Per Cell

This figure was created using an algorithm that examines cells with zero probability adjoining those cells that have non-zero probability of being occupied. The somewhat octagonal shape is a result of the algorithm's calculation of the time required for a target to travel to an adjoining cell. This is a result of an assumption that the target travels to the center of each cell it could occupy. Ideally the shape should be a circular cone, but the octagon seems to be a reasonable approximation. With some refinement, the algorithm could more accurately replicate a circular cone.

We can generate a probability mass distribution from possible cell times by making the probability of occupation proportional to the total possible time of occupation. If we divide each cell's possible time of occupation by the sum of all cells' possible occupation times, the result is a Linear Cone shaped probability distribution. Figure 7 shows the algorithm used to generate the Linear Cone.

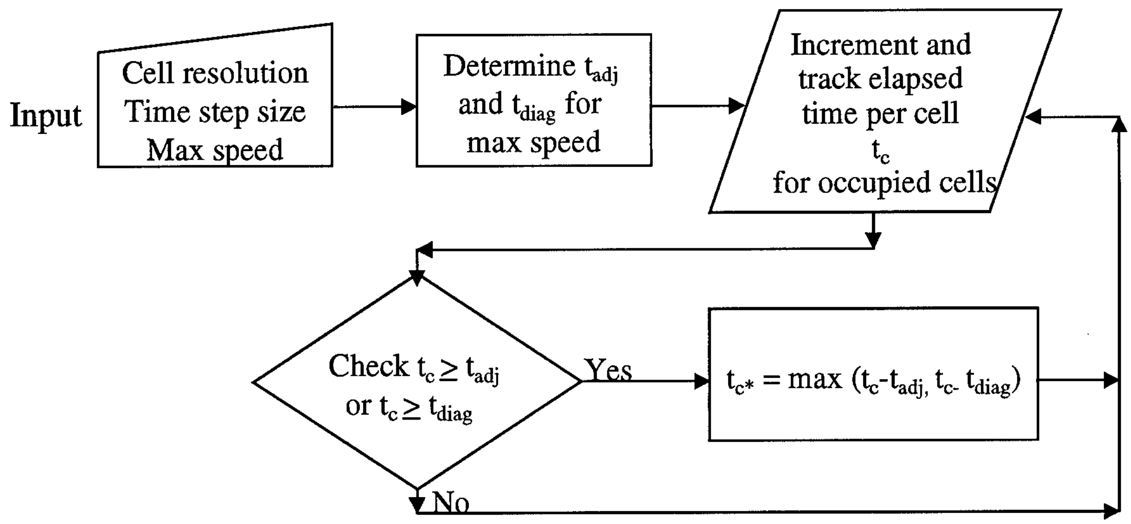

Figure 7: Linear Cone Algorithm

One of the key parts of this algorithm is contained in the second step. During this part of the algorithm, the time required for a target to move to the center of an adjoining cell is computed. This step determines the shortest time needed for the target to travel to adjacent cells (i.e. those cells sharing an edge with another cell) and diagonal cells (i.e. those cells sharing a corner). Travel time is based upon cell size and maximum target speed.

By allowing for travel time in the algorithm, we can incorporate the effects of terrain trafficability. Trafficability values based upon terrain types can lengthen the time required to move between cells. For example, rough or steep terrain may slow a vehicle to half its maximum speed, while a cliff or river could effectively stop movement. Trafficability could be based upon vehicle type or terrain gradient values.

4. RESULTS WITH SELECTED TERRAIN FEATURES

The following examples show how the algorithm can be applied, taking various terrain features into account. In this section of the paper, we investigate various combinations of vehicle speeds and terrain types and their effects on location distribution and information loss. Vehicles with maximum movement speeds of 15, 30, and 45 kilometers per hour are examined on terrain that consists of Go terrain (speed is not degraded), Slow-go terrain (top speed is 60% of maximum), and No-go terrain (top speed is 1% of maximum). The battlefield is divided into a 70 x 70 cell grid, resulting in a maximum entropy value of approximately 8.5. The following discussion presents a summary of effects on information loss due to loss of contact in various terrain, assuming there is only one mobile target on the battlefield.

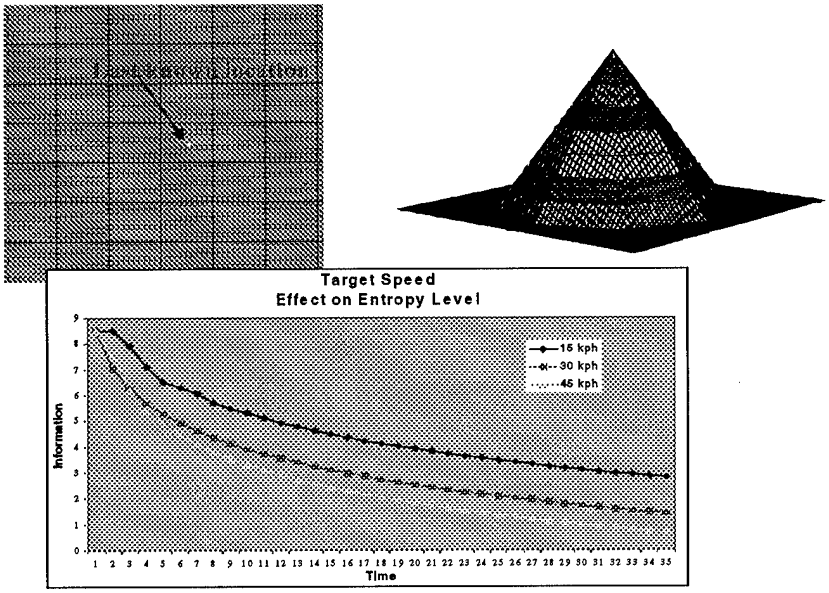

Figure 8: Information loss for go terrain

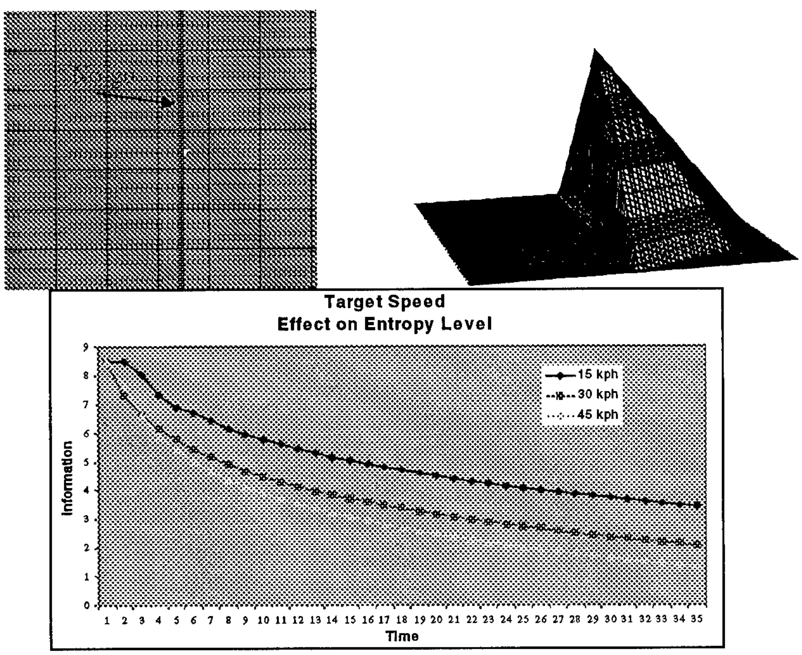

Figure 9: Information loss for target against a river.

a. Go Terrain

Figure 8 shows the effect of target speed on terrain with no trafficability constraints. As expected, the faster a target can travel, the higher the rate of information loss.

b. River

Figure 9 shows results for a scenario where the target was last seen next to a river. In this case, the posterior distributions of target location have a half-cone shape due to

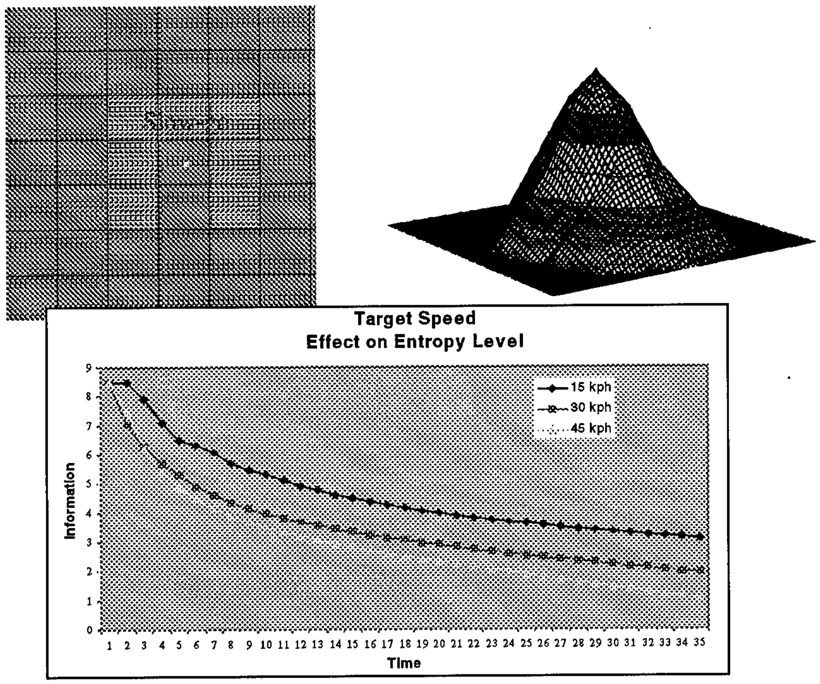

Figure 10: Information loss for target in a box canyon.

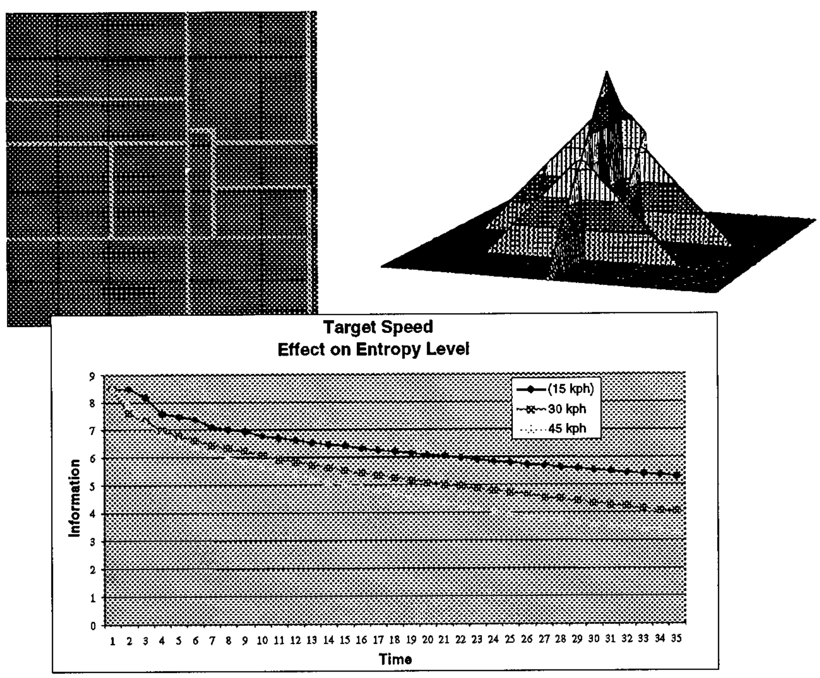

Figure 11: Target motion constrained to roads.

the limited movement ability in the direction of the river. Again we see that faster targets cause faster increases in the information loss function.

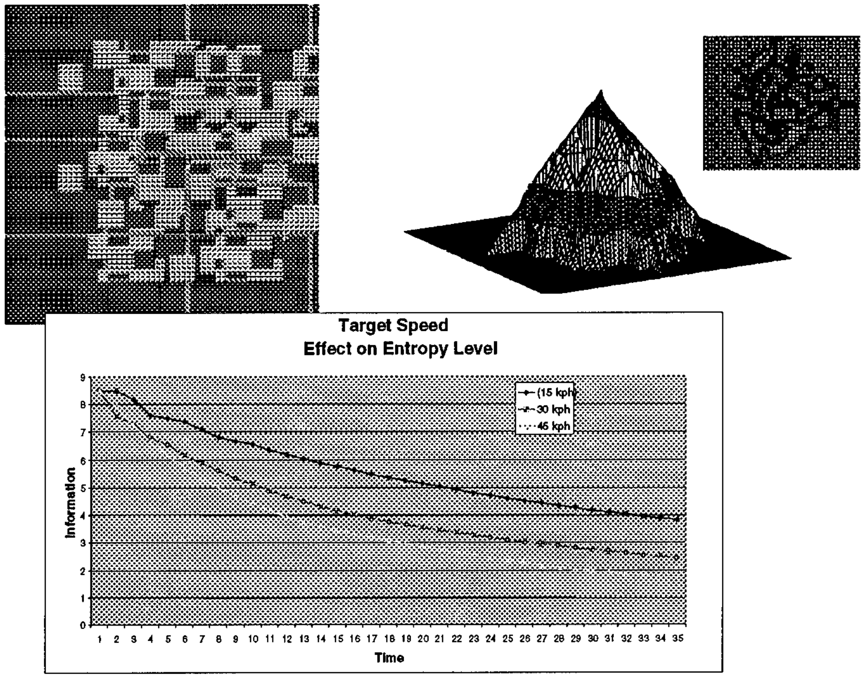

Figure 12: Information loss in a mountainous area.

c. Box Canyon

Figure 10 shows a combination of go and slow-go movement values. The target was last observed on go terrain, surrounded on three sides by slow-go terrain. This scenario may represent a box canyon, in which a vehicle may either climb the sides or travel out the mouth of the canyon. We can see the effects of terrain as the "bulge" of the cone in the location probability distribution.

d. Roads

Figure 11 shows a road network surrounded by no-go terrain. In this case, the probability distribution of target location is concentrated on the roads. Although a vehicle could travel from its original location a distance analogous to that for all go terrain, the information loss is much lower for this case due to the limited number of cells that could possibly be occupied.

e. Mountainous Terrain

Figure 12 contains a random mixture of go, slow-go, and no-go terrain. This case shows the ability of the algorithm to create a probability distribution for a complicated piece of terrain, perhaps a mountainous area that has a network of roads, fields, ponds, and wooded areas.

f. Combined Results

Figure 13 shows combined results of the scenarios discussed above. Also plotted on this graph is the effect of using the Uniform Square model as discussed earlier in this paper. We can see that the Uniform Square model, if used, would tend to overestimate the rate of information loss for all scenarios. Additionally, we can see the wide range of entropy values that result from these scenarios. This leads to the conclusion that we cannot choose a model to represent information gain without regard to vehicle movement speed or terrain type.

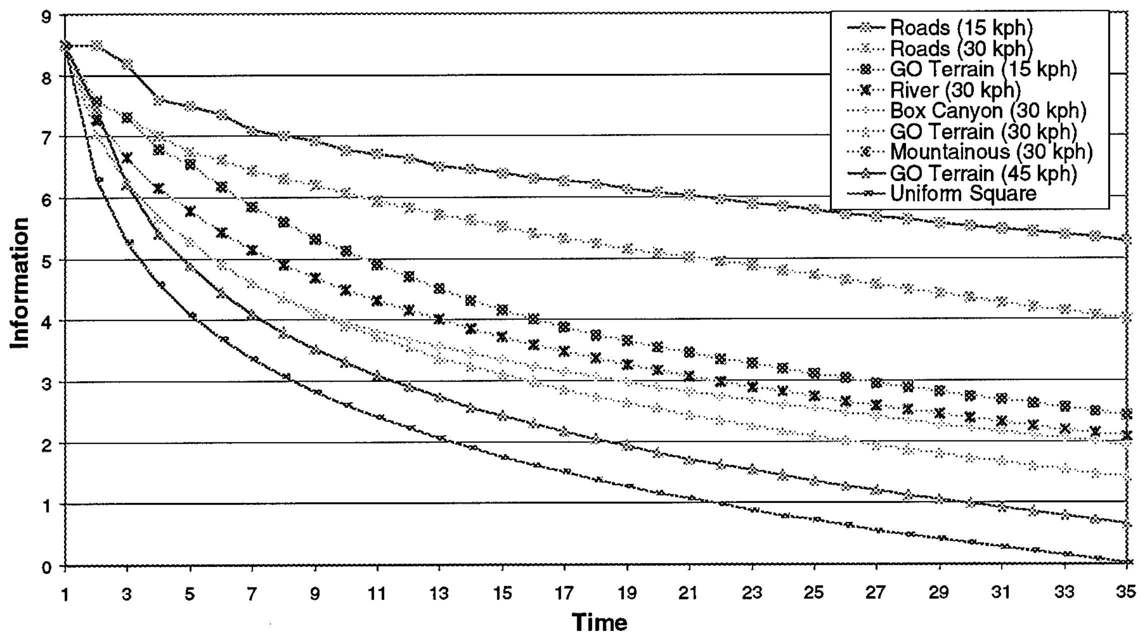

Figure 13: Information loss summary for several terrain and target speeds.

5. CONCLUSIONS

The information gain measure appears to facilitate assessment of performance of battle information systems and to allow assessment of potential effects on situation awareness of factors such as movement of enemy units. Under somewhat crude assumptions about the stochastic movement behavior of an enemy unit, some effects of its mobility on information have been evaluated. Plotted against time, information loss provides insights into effects of changes in movement parameters such as target speed and terrain features.

Much work remains to be done in this area, in our opinion. Further development is needed in the following areas:

- Refine the target movement model and further investigate impacts of various parameters on information loss.
- Develop a two-sided model, which plays Blue's information versus Red's information through time. It appears useful to use an "information ratio" to measure the relative performance of each side's information gain and denial of loss to the enemy. Roughly, such a ratio, viewed from Blue's perspective, would measure information gained over information given away.
- Investigate the effects of possible movement of targets into areas previously swept by Blue's sensors.
- Include effects of information about enemy intent.
- In connection with implementation in combat simulations, provide automatic generation of initial prior distributions, using terrain data and other characteristics of the battle area and enemy force. This should also allow automatic computation of terrain trafficability data for use in Bayesian updating.

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APPENDIX A: NECESSARY CONDITIONS FOR CIRCULAR UNIFORM DISTRIBUTION

Consider "circular uniform" model in polar coordinates. We know, in Cartesian coordinates, the uniform density is of the form $f_{x,y}(x,y) = \dfrac{1}{\pi\tau^2}$; for $x^2 + y^2 \le \tau^2$.

Transformation from (x, y) to (R, Θ):

We know the transformation $R = \sqrt{x^2+y^2}$; $\Theta = \tan^{-1}\left(y/x \right)$ has inverse $x = R\cos\Theta$, $y = R\sin\Theta$,

$$f_{R,\Theta}(r,\theta) = f_{x,y}(r\cos\theta, r\sin\theta)\left|r\right| = \frac{1}{\pi\tau^2}\cdot r$$

$$= \left(\frac{1}{2\pi}\right)\left(\frac{2r}{\tau^2}\right) = f_\Theta(\theta)\cdot f_R(r)$$

where $0 \le \theta \le 2\pi$, $0 \le r \le \tau$.

So, we see $\Theta \sim$ Uniform(0, 2π) and R ~ Triangular, and so Θ and R are independent.

APPENDIX B: DERIVATION OF TARGET LOCATION DISTIBUTION MODEL

Consider independent continuous random variables S, T, and Θ, with distributions S ~ Tri (0, a, a), T ~ U (0, 1), and $\Theta \sim$ U (0, 2π). The probability density functions for these distributions are

$$f_S(s) = \frac{2s}{a^2},\ f_T(t) = 1, \text{ and}$$

$$f_\Theta(\theta) = \frac{1}{2\pi}.$$

Let D = ST, then the probability density function of D is:

$$f_D(d^*) = \int_{-\infty}^{\infty} f_S(s) f_T\left(d^*/s\right)\frac{1}{|s|}ds$$

$$= \int_{d^*}^{a}\frac{2s}{a^2|s|}ds = \frac{2}{a} - \frac{2d^*}{a^2}.$$

If L = DΘ, then in polar coordinates:

$$f_{\Theta,D}(\theta,d^*) = \frac{1}{a\pi}\left[1 - \frac{d^*}{a}\right]$$

where $0 \le \theta \le 2\pi$, $0 \le d^* \le a$.

In rectangular coordinates:

$$f_{X,Y}(x,y) = f_{\Theta,D}(\tan^{-1}\frac{y}{x}, \sqrt{x^2+y^2})\frac{1}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{a\pi}\left[\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{a}\right].$$

The marginal distribution of x is

$$f_X(x) = \int_{-\sqrt{x^2+y^2}}^{\sqrt{x^2+y^2}} \frac{1}{a\pi}\left[\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{a}\right]$$

$$= \frac{2}{a\pi}\left[\ln\frac{\sqrt{a^2-x^2}+a}{|x|} - \frac{\sqrt{a^2-x^2}}{a}\right];$$

where $-a \le x \le a$.